Algebra Review

0.1 Monomial Factors

Factor as indicated:

(a) $3x^4 + 4x^3 - x^2 = x^2(\quad)$

(b) $2\sqrt{x} + 6x^{3/2} = 2\sqrt{x}(\quad)$

(c) $e^{-x} - xe^{-x} + 2x^2e^{-x} = e^{-x}(\quad)$

(d) $x^{-1} - 2 + x = x^{-1} = x^{-1}(\quad)$

(e) $\frac{x}{2} - 6x^2 = \frac{x}{2}(\quad)$

(f) $\sin x + \tan x = \sin x(\quad)$

(g) $\frac{1}{2x^2 + 4x} = \frac{1}{2x}(\quad)$

Solution:

(a) $3x^4 + 4x^3 - x^2 = x^2(3x^2 + 4x - 1)$

(b) $2\sqrt{x} + 6x^{3/2} = 2\sqrt{x}(1 + 3x)$

(c) $e^{-x} - xe^{-x} + 2x^2e^{-x} = e^{-x}(1 - x + 2x^2)$

(d) $x^{-1} - 2 + x = x^{-1} = x^{-1}(1 - 2x + x^2)$

(e) $\frac{x}{2} - 6x^2 = \frac{x}{2}(1 - 12x)$

(f) $\sin x + \tan x = \sin x + \frac{\sin x}{\cos x} = \sin x\left(1 + \frac{1}{\cos x}\right)$

$= \sin x(1 + \sec x)$

(g) $\frac{1}{2x^2 + 4x} = \frac{1}{2x}\left(\frac{1}{x + 2}\right)$

0.2 Binomial Factors

Factor as indicated:

(a) $(x - 1)^2(x) - (x - 1) = (x - 1)(\quad)$

(b) $3(x^2 + 4)(x^2 + 1) + 6(x^2 + 4)^2 = 3(x^2 + 4)(\quad)$

(c) $\sqrt{x^2 + 1} - \frac{x^2}{\sqrt{x^2 + 1}} = \frac{1}{\sqrt{x^2 + 1}}(\quad)$

(d) $(x - 3)^3(x + 2) - 2(x - 3)^2(x + 2)^2 = (x - 3)^2(x + 2)(\quad)$

(e) $(2x + 1)^{3/2}(x^{1/2}) + (2x + 1)^{5/2}(x^{-1/2}) = (2x + 1)^{3/2}(x^{-1/2})(\quad)$

Solution:

(a) $(x - 1)^2(x) - (x - 1) = (x - 1)[(x - 1)x - 1]$

$= (x - 1)(x^2 - x - 1)$

(b) $3(x^2 + 4)(x^2 + 1) + 6(x^2 + 4)^2 = 3(x^2 + 4)[(x^2 + 1) + 2(x^2 + 4)]$

$= 3(x^2 + 4)(3x^2 + 9)$

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(c) $\sqrt{x^2+1} - \dfrac{x^2}{\sqrt{x^2+1}} = (x^2+1)^{1/2} - x^2(x^2+1)^{-1/2}$

$= (x^2+1)^{-1/2}[(x^2+1) - x^2]$

$= \dfrac{1}{\sqrt{x^2+1}}$

(d) $(x-3)^3(x+2) - 2(x-3)^2(x+2)^2 = (x-3)^2(x+2)[(x-3) - 2(x+2)]$

$= (x-3)^2(x+2)(-x-7)$

(e) $(2x+1)^{3/2}(x^{1/2}) + (2x+1)^{5/2}(x^{-1/2}) = (2x+1)^{3/2}(x^{-1/2})[x + (2x+1)]$

$= (2x+1)^{3/2}(x^{-1/2})(3x+1)$

0.3 Factoring Quadratic Expressions

Factor as indicated:

(a) $x^2 - 3x + 2 = (\quad)(\quad)$

(b) $x^2 - 9 = (\quad)(\quad)$

(c) $x^2 + 5x - 6 = (\quad)(\quad)$

(d) $x^2 + 5x + 6 = (\quad)(\quad)$

(e) $2x^2 + 5x - 3 = (\quad)(\quad)$

(f) $e^{2x} + 2 + e^{-2x} = (\quad)^2$

(g) $x^4 - 7x^2 + 12 = (\quad)(\quad)(\quad)$

(h) $1 - \sin^2 x = (\quad)(\quad)$

Solution:

(a) $x^2 - 3x + 2 = (x-2)(x-1)$

(b) $x^2 - 9 = (x+3)(x-3)$

(c) $x^2 + 5x - 6 = (x+6)(x-1)$

(d) $x^2 + 5x + 6 = (x+2)(x+3)$

(e) $2x^2 + 5x - 3 = (2x-1)(x+3)$

(f) $e^{2x} + 2 + e^{-2x} = (e^x + e^{-x})^2$

(g) $x^4 - 7x^2 + 12 = (x^2-3)(x^2-4) = (x^2-3)(x+2)(x-2)$

(h) $1 - \sin^2 x = (1 + \sin x)(1 - \sin x)$

0.4 Cancellation

Reduce each expression to lowest terms:

(a) $\dfrac{3x+9}{6x}$

(b) $\dfrac{x^2}{x^{1/2}}$

(c) $\dfrac{(x+1)^3(x-2) + 3(x+1)^2}{(x+1)^4}$

(d) $\dfrac{x^{1/2} - x^{1/3}}{x^{1/6}}$

(e) $\dfrac{\sqrt{x-1} + (x-1)^{3/2}}{\sqrt{x-1}}$

(f) $\dfrac{1 - (\sin x + \cos x)^2}{2 \sin x}$

Solution:

(a) $\dfrac{3x+9}{6x} = \dfrac{3(x+3)}{3(2x)} = \dfrac{x+3}{2x}$

(b) $\dfrac{x^2}{x^{1/2}} = \dfrac{(x^{1/2})(x^{3/2})}{x^{1/2}} = x^{3/2}$

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(c) $\dfrac{(x+1)^3(x-2)+3(x+1)^2}{(x+1)^4} = \dfrac{(x+1)^2[(x+1)(x-2)+3]}{(x+1)^4}$

$$= \frac{x^2-x+1}{(x+1)^2}$$

(d) $\dfrac{x^{1/2}-x^{1/3}}{x^{1/6}} = \dfrac{x^{1/6}(x^{2/6}-x^{1/6})}{x^{1/6}} = x^{1/3}-x^{1/6}$

(e) $\dfrac{\sqrt{x-1}+(x-1)^{3/2}}{\sqrt{x-1}} = \dfrac{\sqrt{x-1}[1+(x-1)]}{\sqrt{x-1}} = x$

(f) $\dfrac{1-(\sin x+\cos x)^2}{2\sin x} = \dfrac{1-(\sin^2 x+2\sin x\cos x+\cos^2 x)}{2\sin x}$

$$= \frac{1-(\sin^2 x+\cos^2 x)-2\sin x\cos x}{2\sin x}$$

$$= \frac{1-1-2\sin x\cos x}{2\sin x} = -\cos x$$

0.5 Quadratic Formula

	Equation	*Solve for*
(a)	$x^2-4x-1=0$	x
(b)	$2x^2+x-3=0$	x
(c)	$\cos^2 x+3\cos x+2=0$	$\cos x$
(d)	$x^2-xy-(1+y^2)=0$	x
(e)	$x^4-4x^2+2=0$	x^2

Solution:

(a) $x = \dfrac{4\pm\sqrt{16+4}}{2} = \dfrac{4\pm\sqrt{20}}{2} = \dfrac{4\pm 2\sqrt{5}}{2} = 2\pm\sqrt{5}$

(b) $x = \dfrac{-1\pm\sqrt{1+24}}{4} = \dfrac{-1\pm 5}{4}$

$x = \dfrac{4}{4} = 1$ or $x = -\dfrac{6}{4} = -\dfrac{3}{2}$

(c) $\cos x = \dfrac{-3\pm\sqrt{9-8}}{2} = \dfrac{-3\pm 1}{2}$

$\cos x = -\dfrac{2}{2} = -1$ or $\cos x = -\dfrac{4}{2} = -2$

(d) $x = \dfrac{y\pm\sqrt{y^2+4(1+y^2)}}{2} = \dfrac{y\pm\sqrt{y^2+4+4y^2}}{2}$

$$= \frac{y\pm\sqrt{5y^2+4}}{2}$$

(e) $x^2 = \dfrac{4\pm\sqrt{16-8}}{2} = \dfrac{4\pm\sqrt{8}}{2} = \dfrac{4\pm 2\sqrt{2}}{2} = 2\pm\sqrt{2}$

0.6 Synthetic Division

Using synthetic division to factor as indicated:

(a) $x^3 - 4x^2 + 2x + 1 = (x - 1)(\quad)$

(b) $2x^3 + 5x + 7 = (x + 1)(\quad)$

(c) $x^4 - 3x^3 + x^2 + x + 2 = (x - 2)(\quad)$

(d) $4x^4 + 3x^2 - 1 = (2x - 1)(\quad)$

Solution:

(a) $x^3 - 4x^2 + 2x + 1$

$$\begin{array}{r|rrrr} 1 & 1 & -4 & 2 & 1 \\ & & 1 & -3 & -1 \\ \hline & 1 & -3 & -1 & 0 \end{array}$$

$x^3 - 4x^2 + 2x + 1 = (x - 1)(x^2 - 3x - 1)$

(b) $2x^3 + 5x + 7$

$$\begin{array}{r|rrrr} -1 & 2 & 0 & 5 & 7 \\ & & -2 & 2 & -7 \\ \hline & 2 & -2 & 7 & 0 \end{array}$$

$2x^3 - 5x + 7 = (x + 1)(2x^2 - 2x + 7)$

(c) $x^4 - 3x^3 + x^2 + x + 2$

$$\begin{array}{r|rrrrr} 2 & 1 & -3 & 1 & 1 & 2 \\ & & 2 & -2 & -2 & -2 \\ \hline & 1 & -1 & -1 & -1 & 0 \end{array}$$

$x^4 - 3x^3 + x^2 + x + 2 = (x - 2)(x^3 - x^2 - x - 1)$

(d) $4x^4 + 3x^2 - 1$

$$\begin{array}{r|rrrrr} \frac{1}{2} & 4 & 0 & 3 & 0 & -1 \\ & & 2 & 1 & 2 & 1 \\ \hline & 4 & 2 & 4 & 2 & 0 \end{array}$$

$$\begin{aligned} 4x^4 + 3x^2 - 1 &= \left(x - \frac{1}{2}\right)(4x^3 + 2x^2 + 4x + 2) \\ &= (2x - 1)(2x^3 + x^2 + 2x + 1) \end{aligned}$$

0.7 Special Products

Factor completely (into linear or irreducible quadratic factors):

(a) $x^3 - 27$

(b) $x^3 - 3x^2 + 3x - 1$

(c) $x^3 + 6x^2 + 12x + 8$

(d) $x^4 - 25$

(e) $x^4 - 8x^3 + 24x^2 - 32x + 16$

Solution:

(a) $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$

(b) $x^3 - 3x^2 + 3x - 1 = (x - 1)^3$

(c) $x^3 + 6x^2 + 12x + 8 = x^3 + 3(2)x^2 + 3(2^2)x + 2^3 = (x + 2)^3$

(d) $x^4 - 25 = (x^2 + 5)(x^2 - 5) = (x^2 + 5)\left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)$

(e) $x^4 - 8x^3 + 24x^2 - 32x + 16 = x^4 - 4(2)x^3 + 6(2^2)x^2 - 4(2^3)x + 2^4 = (x - 2)^4$

0.8 Factoring by Grouping

Factor completely (into linear or irreducible quadratic factors):

(a) $x^3 + 4x^2 - 2x - 8$

(b) $x^3 + 2x^2 + 3x + 6$

(c) $5\cos^2 x - 5\sin^2 x + \sin x + \cos x$

(d) $\cos^2 x + 4\cos x + 4 - \tan^2 x$

Solution:

(a)
$$\begin{aligned} x^3 + 4x^2 - 2x - 8 &= x^2(x + 4) - 2(x + 4) \\ &= (x^2 - 2)(x + 4) \\ &= \left(x + \sqrt{2}\right)\left(x - \sqrt{2}\right)(x + 4) \end{aligned}$$

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(b) $x^3 + 2x^2 + 3x + 6 = x^2(x + 2) + 3(x + 2)$

$= (x^2 + 3)(x + 2)$

(c) $5\cos^2 x - 5\sin^2 x + \sin x + \cos x = 5(\cos^2 x - \sin^2 x) + (\sin x + \cos x)$

$= 5(\cos x - \sin x)(\cos x - \sin x) + (\cos x + \sin x)$

$= (\cos x + \sin x)[5(\cos x - \sin x) + 1]$

(d) $\cos^2 x + 4\cos x + 4 - \tan^2 x = (\cos x + 2)^2 - \tan^2 x$

$= (\cos x + 2 + \tan x)(\cos x + 2 - \tan x)$

0.9 Simplifying

Rewrite each of the following in simplest form:

(a) $\dfrac{(x-1)(x+3)-(x+1)^2}{x+1}$

(b) $\dfrac{\sqrt{x^2+1} - \dfrac{1}{\sqrt{x^2+1}}}{x^2+1}$

(c) $\dfrac{x^2-5x+6}{x^2-4x+4}$

(d) $\dfrac{1}{x+1} - \dfrac{1}{x-1} - \dfrac{2}{x^2-1}$

(e) $\dfrac{x(-2x)}{2\sqrt{1-x^2}} + \sqrt{1-x^2} + \dfrac{1}{\sqrt{1-x^2}}$

Solution:

(a) $\dfrac{(x-1)(x+3)-(x+1)^2}{x+1} = \dfrac{(x^2+2x-3)-(x^2+2x+1)}{x+1} = \dfrac{-4}{x+1}$

(b) $\dfrac{\sqrt{x^2+1} - \dfrac{1}{\sqrt{x^2+1}}}{x^2+1} = \dfrac{\dfrac{1}{\sqrt{x^2+1}}(x^2+1-1)}{x^2+1} = \dfrac{x^2+1-1}{\sqrt{x^2+1}(x^2+1)} = \dfrac{x^2}{(x^2+1)^{3/2}}$

(c) $\dfrac{x^2-5x+6}{x^2-4x+4} = \dfrac{(x-2)(x-3)}{(x-2)^2} = \dfrac{x-3}{x-2}$

(d) $\dfrac{1}{x+1} - \dfrac{1}{x-1} - \dfrac{2}{x^2-1} = \dfrac{(x-1)-(x+1)-2}{x^2-1} = \dfrac{-4}{x^2-1}$

(e) $\dfrac{x(-2x)}{2\sqrt{1-x^2}} + \sqrt{1-x^2} + \dfrac{1}{\sqrt{1-x^2}} = \dfrac{-x^2}{\sqrt{1-x^2}} + \dfrac{1-x^2}{\sqrt{1-x^2}} + \dfrac{1}{\sqrt{1-x^2}} = \dfrac{2-2x^2}{\sqrt{1-x^2}}$

$= \dfrac{2(1-x^2)}{\sqrt{1-x^2}} = 2\sqrt{1-x^2}$

0.10 Rationalizing

Remove the sum or difference from the denominator by multiplying the numerator and denominator by the conjugate of the denominator.

(a) $\dfrac{1}{1-\cos x}$

(b) $\dfrac{x}{1-\sqrt{x^2+1}}$

(c) $\dfrac{2}{x+\sqrt{x^2+1}}$

Solution:

(a) $\dfrac{1}{1-\cos x} = \left(\dfrac{1}{1-\cos x}\right)\left(\dfrac{1+\cos x}{1+\cos x}\right)$

$= \dfrac{1+\cos x}{1-\cos^2 x} = \dfrac{1+\cos x}{\sin^2 x}$

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(b) $$\left(\frac{x}{1-\sqrt{x^2+1}}\right)\left(\frac{1+\sqrt{x^2+1}}{1+\sqrt{x^2+1}}\right)=\frac{x\left(1+\sqrt{x^2+1}\right)}{1-(x^2+1)}$$

$$=\frac{x\left(1+\sqrt{x^2+1}\right)}{-x^2}=\frac{1+\sqrt{x^2+1}}{-x}$$

(c) $$\left(\frac{2}{x+\sqrt{x^2+1}}\right)\left(\frac{x-\sqrt{x^2+1}}{x-\sqrt{x^2+1}}\right)=\frac{2\left(x-\sqrt{x^2+1}\right)}{x^2-(x^2+1)}=-2\left(x-\sqrt{x^2+1}\right)$$

0.11 Algebraic Errors to Avoid

Error	*Correct form*	*Comments*
$a-(x-b)\neq a-x-b$	$a-(x-b)=a-x+b$	Change all signs when distribution negative through parentheses.
$(a+b)^2\neq a^2+b^2$	$(a+b)^2=a^2+2ab+b^2$	Don't forget middle term when squaring binomials.
$\left(\frac{1}{2}a\right)\left(\frac{1}{2}b\right)\neq\frac{1}{2}ab$	$\left(\frac{1}{2}a\right)\left(\frac{1}{2}b\right)=\frac{1}{4}(ab)$	1/2 occurs twice as a factor.
$\frac{a}{x+b}\neq\frac{a}{x}+\frac{a}{b}$	Leave as $\frac{a}{x+b}$	Don't add denominators when adding fractions.
$\frac{1}{a}+\frac{1}{b}\neq\frac{1}{a+b}$	$\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}$	Use definition for adding fractions.
$\frac{\frac{x}{a}}{b}\neq\frac{bx}{a}$	$\frac{\frac{x}{a}}{b}=\left(\frac{x}{a}\right)\left(\frac{1}{b}\right)=\frac{x}{ab}$	Multiply by reciprocal of the denominator.
$\frac{1}{3x}\neq\frac{1}{3}x$	$\frac{1}{3x}=\frac{1}{3}\cdot\frac{1}{x}$	Use definition for multiplying fractions.
$1/x+2\neq\frac{1}{x+2}$	$1/x+2=\frac{1}{x}+2$	Be careful when using a slash to denote division.
$(x^2)^3\neq x^5$	$(x^2)^3=x^{2\cdot3}=x^6$	Multiply exponents when an exponential form is raised to a power.
$2x^3\neq(2x)^3$	$2x^3=2(x^3)$	Exponents have priority over coefficients.
$\frac{1}{x^2+x^3}\neq x^{-2}+x^{-3}$	Leave as $\frac{1}{x^2+x^3}$	Don't shift term-by-term from denominator to numerator.
$\sqrt{5x}\neq5\sqrt{x}$	$\sqrt{5x}=\sqrt{5}\sqrt{x}$	Radicals apply to every factor inside radical.
$\sqrt{x^2+a^2}\neq x+a$	Leave as $\sqrt{x^2+a^2}$	Don't apply radicals term-by-term.
$\frac{a+bx}{a}\neq1+bx$	$\frac{a+bx}{a}=1+\frac{b}{a}x$	Cancel common factor, *not* common terms.
$\frac{a+ax}{a}\neq a+x$	$\frac{a+ax}{a}=1+x$	Factor *before* canceling.